SOLAR STRUCTURE IN TERMS OF GAUSS' HYPERGEOMETRIC FUNCTION

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Abstract. Hydrostatic equilibrium and energy conservation determine the conditions in the gravitationally stabilized solar fusion reactor. We assume a matter density distribution varying non-linearly through the central region of the Sun. The analytic solutions of the differential equations of mass conservation, hydrostatic equilibrium, and energy conservation, together with the equation of state of the perfect gas and a nuclear energy generation rate $\epsilon = \epsilon_0 \rho^n T^m$, are given in terms of Gauss' hypergeometric function. This model for the structure of the Sun gives the run of density, mass, pressure, temperature, and nuclear energy generation through the central region of the Sun. Because of the assumption of a matter density distribution, the conditions of hydrostatic equilibrium and energy conservation are separated from the mode of energy transport in the Sun.

1 Hydrostatic Equilibrium

In the following we are concerned with the hydrostatic equilibrium of the purely gaseous spherical central region of the Sun generating energy by nuclear reactions at a certain rate (Chandrasekhar, 1939/1957; Stein, 1966). For this gaseous sphere we assume that the matter density varies non-linearly from the center outward, depending on two parameters δ and γ,

$$\rho(x) = \rho_c f_D(x), \tag{1}$$

$$f_D(x) = [1 - x^\delta]^\gamma, \tag{2}$$

where x denotes the dimensionless distance variable, $x = r/R_\odot, 0 \leq x \leq 1$, $R_\odot$ is the solar radius, $\delta > 0, \gamma > 0$ and γ is kept a positive integer in the following considerations. The choice of the density distribution in (1) and (2) reveals immediately that $\rho(x = 0) = \rho_c$ is the central density of the configuration and $\rho(x = 1) = 0$ is a boundary condition for hydrostatic equilibrium of the gaseous configuration. For the range $0 \leq x \leq 0.3$ the density distribution in (1) and (2) can be fit numerically to computed data for solar models by chosing $\delta = 1.28$ and $\gamma = 10$ (Haubold and Mathai 1994). For these values of δ and γ the function $f_D(x)$ in (2) is shown in Figure 1. The choice of restricting x to $x \leq 0.3$ is justified by looking at a Standard Solar Model (Bahcall and Pinsonneault, 1992) which shows that $x \leq 0.3$ comprises what is considered to be the gravitationally stabilized solar fusion reactor. More precisely, 95% of the solar luminosity is produced within the region $x < 0.2 (M < 0.3 M_\odot)$. The half-

peak value for the matter density occurs at $x = 0.1$ and the half-peak value for the temperature occurs at $x = 0.25$. The region $x \leq 0.3$ is also the place where the solar neutrino fluxes are generated. As we are concerned with a spherically symmetrical distribution of matter, the mass $M(x)$ within the radius x having the density distribution given in (1) and (2) is

$$M(x) = M_{\odot} f_M(x), \tag{3}$$

$$f_M(x) = \frac{(\frac{3}{\delta}+1)(\frac{3}{\delta}+2)\cdots(\frac{3}{\delta}+\gamma)}{\gamma!} x^3 \, {}_2F_1(-\gamma, \frac{3}{\delta}; \frac{3}{\delta}+1; x^{\delta}), \tag{4}$$

where $M_{\odot}$ denotes the solar mass and ${}_2F_1(.)$ is Gauss' hypergeometric function (see, for example, Mathai, 1993). Equations (3) and (4) are satisfying the boundary condition $M(x = 0) = 0$ and determine the central value ρ_c of the matter density through the boundary condition $M(x = 1) = M_{\odot}$, where ρ_c depends then only on δ and γ of the chosen density distribution in (1) and (2). The function $f_M(x)$ in (4) is shown in Figure 2, using Mathematica (Wolfram, 1991).

For hydrostatic equilibrium of the gaseous configuration the internal pressure needs to balance the gravitational attraction. The pressure distribution follows by integration of the respective differential equation for hydrostatic equilibrium, making use of the density distribution in (1) and the mass distribution in (3), that is

$$P(x) = \frac{9}{4\pi} G \frac{M_{\odot}^2}{R_{\odot}^4} f_P(x), \tag{5}$$

$$
\begin{aligned}
f_P(x) \quad &= \quad \left[\frac{(\frac{3}{\delta}+1)(\frac{3}{\delta}+2)\dots(\frac{3}{\delta}+\gamma)}{\gamma!}\right]^2 \frac{1}{\delta^2}\sum_{m=0}^{\gamma}\frac{(-\gamma)_m}{m!(\frac{3}{\delta}+m)(\frac{2}{\delta}+m)} \\
&\times \left[\frac{\gamma!}{(\frac{2}{\delta}+m+1)_\gamma} - x^{\delta m+2}\,{}_2F_1(-\gamma,\frac{2}{\delta}+m;\frac{2}{\delta}+m+1;x^\delta)\right], \quad (6)
\end{aligned}
$$

where G is Newton's constant and ${}_2F_1(.)$ denotes again Gauss' hypergeometric function (Mathai, 1993).

The Pochhammer symbol $(\frac{2}{\delta}+m+1)_\gamma = \Gamma(\frac{2}{\delta}+m+1+\gamma)/\Gamma(\frac{2}{\delta}+m+1)$ often appears in series expansions for hypergeometric functions. Equations (5) and (6) give the value of the pressure P_c at the centre of the gaseous configuration and satisfy the condition $P(x=1)=0$. The graph of the function $f_P(x)$ in (6) is shown in Figure 3,using Mathematica (Wolfram, 1991).

2 Equation of State

It should be noted that $P(x)$ in (5) denotes the total pressure of the gaseous configuration, that is the sum of the gas kinetic pressure and the radiation pressure (according to Stefan-Boltzmann's law)(Chandrasekhar, 1939/1957; Stein, 1966). However, the radiation pressure, although the ratio of radiation pressure to gas pressure increases towards the center of the Sun, remains negligibly small in comparison to the gas kinetic pressure. Thus, Equation (5) can be considered to represent the run of the gas pressure through the configuration under consideration. Further, the matter density is so low that at the temperatures involved the material follows the equation of state of the perfect gas. Therefore,

the temperature distribution throughout the gaseous configuration is given by

$$T(x) = 3\frac{\mu}{kN_A}G\frac{M_\odot}{R_\odot}f_T(x), \tag{7}$$

$$\begin{aligned} f_T(x) \quad &= \quad \left[\frac{(\frac{3}{\delta}+1)(\frac{3}{\delta}+2)\cdots(\frac{3}{\delta}+\gamma)}{\gamma!}\right]\frac{1}{\delta^2}\frac{1}{[1-x^\delta]^\gamma}\sum_{m=0}^{\gamma}\frac{(-\gamma)_m}{m!(\frac{3}{\delta}+m)(\frac{2}{\delta}+m)} \\ &\times\left[\frac{\gamma!}{(\frac{2}{\delta}+m+1)_\gamma} - x^{\delta m+2}\,{}_2F_1(-\gamma,\frac{2}{\delta}+m;\frac{2}{\delta}+m+1;x^\delta)\right], \end{aligned} \tag{8}$$

where k is the Boltzmann constant, N_A Avogadro's number, μ the mean molecular weight, and ${}_2F_1(.)$ Gauss' hypergeometric function (Mathai, 1993). Equations (7) and (8) reveal the central temperature for $T(x=0) = T_c$ and satisfy the boundary condition $T(x=1) = 0$. Since the gas in the central region of the Sun can be treated as completely ionised, the mean molecular weight μ is given by $\mu = (2X + \frac{3}{4}Y + \frac{1}{2}Z)^{-1}$, where X, Y, Z are relative abundances by mass of hydrogen, helium, and heavy elements, respectively, and $X + Y + Z = 1$. The run of the function $f_T(x)$ in (8) is shown in Figure 4, using Mathematica (Wolfram, 1991).

3 Nuclear Energy Generation Rate

Hydrostatic equilibrium and energy conservation are determining the physical conditions in the central part of the Sun. In the preceeding Sections, the run of density, mass, pressure, and temperature have been given for a gaseous configuration in hydrostatic equilibrium based on the equation of state of the perfect gas (Equations (1)-(8)). In the following, a representation for the nuclear energy

generation rate,

$$\epsilon(\rho, T) = \epsilon_0 \rho^n T^m, \tag{9}$$

will be sought which takes into account the above given distributions of density and temperature and which can be used to integrate the differential equation of energy conservation throughout the gaseous configuration considered in Sections 1 and 2. In Equation (9), n denotes the density exponent, m the temperature exponent, and ϵ_0 a positive constant determined by the specific reactions for the generation of nuclear energy. Using the equation of state of the perfect gas, Equation (9) can be rewritten more conveniently,

$$\begin{aligned} \epsilon(x) &= \epsilon_0 \left(\frac{\mu}{kN_A}\right)^m [\rho(x)]^{n-m} [P(x)]^m \\ & \epsilon_0 \left(\frac{\mu}{kN_A}\right)^m \rho_c^{n-m} P_c^m \left[\frac{P(x)}{P_c}\right]^m \left[1 - x^\delta\right]^{\gamma(n-m)}, \end{aligned} \tag{10}$$

where we note that $0 \leq P(x)/P_c \leq 1$, and $P(x)$ is given in (5). Subsequently, we can write P_c as follows:

$$P_c = \frac{9}{4\pi} G \frac{M_\odot^2}{R_\odot^4} \left[\frac{(\frac{3}{\delta}+1)(\frac{3}{\delta}+2)\cdots(\frac{3}{\delta}+\gamma)}{\gamma!}\right]^2 \frac{1}{\delta^2} \eta(\gamma), \tag{11}$$

$$\eta(\gamma) = \sum_{\nu=0}^{\gamma} \frac{(-\gamma)_\nu}{\nu!} \frac{1}{(\frac{2}{\delta}+\nu)(\frac{3}{\delta}+\nu)} \frac{\gamma!}{(\frac{2}{\delta}+\nu+1)\cdots(\frac{2}{\delta}+\nu+\gamma)}. \tag{12}$$

Taking the ratio of the pressure $P(x)$ at the location x to the central value of it one can write

$$\frac{P(x)}{P_c} = 1 - \frac{1}{\eta(\gamma)} x^2 h(x), \tag{13}$$

$$\begin{aligned} h(x) &= \sum_{m_1=0}^{\gamma} \sum_{m_2=0}^{\gamma} \frac{(-\gamma)_{m_1}}{m_1!} \frac{(-\gamma)_{m_2}}{m_2!} \\ & \times \frac{1}{(\frac{3}{\delta}+m_1)(\frac{2}{\delta}+m_1+m_2)} x^{\delta(m_1+m_2)}. \end{aligned} \tag{14}$$

Note that $h(x)$ is a polynomial of degree 2γ in x^δ. Denoting the polynomial $h(x)$ by

$$h(x) = a_0 + a_1[x^\delta] + a_2[x^\delta]^2 + \ldots + a_{2\gamma}[x^\delta]^{2\gamma}, \tag{15}$$

we obtain for (13) with a view to (10),

$$\begin{aligned} \left[\frac{P(x)}{P_c}\right]^m &= \left[1 - \frac{1}{\eta(\gamma)} x^2 h(x)\right]^m \\ &= \sum_{q=0}^{m} \frac{(-m)_q}{q!} \left[\frac{1}{\eta(\gamma)}\right]^q x^{2q} \left[h(x)\right]^q , \end{aligned} \tag{16}$$

where we can expand $[h(x)]^q$ by using a multinomial expansion. That is

$$[h(x)]^q = \sum_{n_0=0}^{q} \sum_{n_1=0}^{q} \cdots \sum_{n_{2\gamma}=0}^{q} \frac{q! a_0^{n_0} a_1^{n_1} \ldots a_{2\gamma}^{n_{2\gamma}}}{n_0! n_1! \ldots n_{2\gamma}!} \left[x^\delta\right]^{n_1 + 2n_2 + \ldots + (2\gamma) n_{2\gamma}} , \tag{17}$$

where $n_0 + n_1 + \ldots + n_{2\gamma} = q$. Note also that since $0 \leq x^\delta \leq 1$ we have for $x < 1$, in Equation (10),

$$\left[1 - x^\delta\right]^{-\gamma(m-n)} = \sum_{s=0}^{\gamma(m-n)} \frac{[\gamma(m-n)]_s}{s!} x^{\delta s}. \tag{18}$$

For the nuclear energy generation rate in (10) we obtain finally

$$\epsilon(x) = \epsilon_0 \rho_c^n T_c^m f x^{\delta s + 2q + \delta[n_1 + 2n_2 + \ldots + (2\gamma) n_{2\gamma}]}, \tag{19}$$

where

$$\begin{aligned} f &= f(\delta, \gamma, m, n; s, q, n_0, n_1, \ldots, n_{2\gamma}; a_0, a_1, \ldots, a_{2\gamma}) \\ &= \sum_{s=0}^{\gamma(m-n)} \frac{[\gamma(m-n)]_s}{s!} \sum_{q=0}^{m} \frac{(-m)_q}{q!} \left(\frac{1}{\eta(\gamma)}\right)^q \\ &\times \sum_{n_0=0}^{q} \cdots \sum_{n_{2\gamma}=0}^{q} \frac{q! a_0^{n_0} a_1^{n_1} \ldots a_{2\gamma}^{n_{2\gamma}}}{n_0! n_1! \ldots n_{2\gamma}!}, \end{aligned} \tag{20}$$

with $n_0 + n_1 + \ldots + n_{2\gamma} = q$.

Equation (19) reflects the analytic representation of the nuclear energy generation rate in (9) taking into account the run of physical quantities given for the gaseous configuration in hydrostatic equilibrium considered in Sections 1 and 2. In deriving the explicit dependence of ϵ on the density and temperature we have exercised particular care, because the rate of nuclear energy production is very highly temperature sensitive. Small changes in the temperature in the central part of the Sun are adequate to balance large differences in luminosity.

4 Total Nuclear Energy Generation

The total net rate of nuclear energy generation is equal to the luminosity of the Sun, that means the generation of energy by nuclear reactions in the central part of the Sun has to continually replenish that energy radiated away at the surface. If $L(x)$ denotes the outflow of energy across the spherical surface at distance x from the center, then in equilibrium the average energy production at distance x is

$$L(x) = 4\pi R_{\odot}^3 \int_0^x dt t^2 \rho(t) \epsilon(t), \tag{21}$$

where $\rho(x)$ and $\epsilon(x)$ are given in (1) and (19), respectively (Chandrasekhar, 1939/1957; Stein, 1966). Collecting all factors containing the relative distance variable x, the integral to be evaluated is the following, denoting it by $g(x)$,

$$\begin{aligned} g(x) &= R_\odot^3 \int_0^x dt t^2 \rho(t) t^{\delta s+2q+\delta[n_1+2n_2+\ldots+(2\gamma)n_{2\gamma}]} \\ &= \rho_c R_\odot^3 \int_0^x dt t^2 [1-t^\delta]^\gamma t^{\delta s+2q+\delta[n_1+2n_2+\ldots+(2\gamma)n_{2\gamma}]} \\ &= \rho_c R_\odot^3 \frac{1}{\delta} \int_0^{x^\delta} dv (1-v)^\gamma v^{s+\frac{1}{\delta}(3+2q)+[n_1+2n_2+\ldots+(2\gamma)n_{2\gamma}]-1} \end{aligned} \tag{22}$$

by setting $x^\delta = v$. Equation (22) represents an incomplete beta function which can be written as a series or in terms of a hypergeometric function. Let $s^* = s + \frac{1}{\delta}(3+2q) + n_1 + 2n_2 + \ldots + (2\gamma)n_{2\gamma}$, then we have

$$\begin{aligned} g(x) &= \rho_c R_\odot^3 \frac{1}{\delta} \int_0^{x^\delta} dv (1-v)^\gamma v^{s^*-1} \\ &= \rho_c R_\odot^3 \frac{1}{\delta} \sum_{l=0}^{\gamma} \frac{(-\gamma)_l}{l!} \int_0^{x^\delta} dv v^{s^*+l-1} \\ &= \rho_c R_\odot^3 \frac{1}{\delta} \sum_{l=0}^{\gamma} \frac{(-\gamma)_l}{l!} \frac{[x^\delta]^{l+s^*}}{l+s^*} \\ &= \rho_c R_\odot^3 \frac{1}{\delta} [x^\delta]^{s^*} \sum_{l=0}^{\gamma} \frac{(-\gamma)_l}{l!} \frac{[x^\delta]^l}{s^*+l} \\ &= \rho_c R_\odot^3 \frac{1}{\delta s^*} [x^\delta]^{s^*} {}_2F_1(-\gamma, s^*; s^*+1; x^\delta), \end{aligned} \tag{23}$$

where ${}_2F_1(.)$ is Gauss' hypergeometric function (Mathai, 1993). Hence

$$L(x) = 4\pi\epsilon_0 \rho_c^{n+1} T_c^m R_\odot^3 \frac{1}{\delta} f \frac{1}{s^*} [x^\delta]^{s^*} {}_2F_1(-\gamma, s^*; s^*+1; x^\delta), \tag{24}$$

where f is defined in (20). The luminosity, $L(x=1) = L_\odot$, is given by

$$L_\odot = 4\pi\epsilon_0 \rho_c^{n+1} T_c^m R_\odot^3 \frac{1}{\delta} f \frac{1}{s^*} \frac{\gamma!}{(s^*+1)(s^*+2)\ldots(s^*+\gamma)}, \tag{25}$$

using the relation ${}_2F_1(a,b;c;1) = [\Gamma(c)\Gamma(c-a-b)]/[\Gamma(c-a)\Gamma(c-b)]$ (Mathai, 1993).

References

Bahcall, J.N. and Pinsonneault, H.M.: 1992, Rev. Mod. Phys. 64,885.

Chadrasekhar, S.: 1939/1957, An Introduction to the Study of Stellar Structure, Dover Publications Inc., New York.

Haubold, H.J. and Mathai, A.M.: 1994, in H.J. Haubold and L.I. Onuora (eds.), 'Basic Space Science', AIP Conference Proceedings Vol. 320, American Institute of Physics, New York.

Mathai, A.M.: 1993, A Handbook of Generalized Special Functions for Statistical and Physical Sciences, Clarendon Press, Oxford.

Stein, R.F.: 1966, in R.F. Stein and A.G.W. Cameron (eds.), 'Stellar Evolution', Plenum Press, New York.

Wolfram, S.: 1991, Mathematica - A System for Doing Mathematics by Computer, Addison-Wesley Publishing Company Inc., Redwood City.